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A THERMOMECHANICAL CONSTITUTIVE THEORY FOR ELASTIC COMPOSITES WITH DISTRIBUTED DAMAGE—I. THEORETICAL DEVELOPMENT

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Abstract—A continuum mechanics approach is utilized herein to develop a model for predicting the thermomechanical constitution of elastic composites subjected to both monotonic and cyclic fatigue loading. In this model the damage is characterized by a set of second-order tensor valued internal state variables representing locally averaged measures of specific damage states such as matrix cracks, fiber–matrix debonding, interlaminar cracking, or any other damage state. Locally averaged history dependent constitutive equations are posed utilizing constraints imposed from thermodynamics with internal state variables. In Part I the thermodynamics with internal state variables is constructed and it is shown that suitable definitions of the locally averaged field variables will lead to useful thermodynamic constraints on a local scale containing statistically homogeneous damage. Based on this result the Helmholtz free energy is then expanded in a Taylor series in terms of strain, temperature, and the internal state variables to obtain the stress–strain relation for composites with damage. In Part II the three-dimensional tensor equations developed in Part I are simplified using material symmetry constraints and are written in engineering notation. The resulting constitutive model is then cast into laminate equations and an example problem is solved and compared to experimental results. It is concluded that although the model requires further development and extensive experimental verification it may be a useful tool in characterizing the thermomechanical constitutive behavior of continuous fiber composites with damage.

INTRODUCTION

A model for predicting the effect of microstructural damage on the constitutive behavior of continuous fiber-reinforced laminated composites is presented in this two part paper. In Part I, the general model is developed from a theoretical treatment of damage mechanics using continuum mechanics and thermodynamic principles. In Part II, the constitutive model is specialized for the case of matrix crack damage confined to the 90° plies of cross-ply laminates. Predicted values of the damage-degraded axial modulus of cross-ply laminates with a variety of stacking sequences are compared to experimental values.

While the motivation for the research is to model laminated composites, the general model formulated in Part I is applicable to a broad class of media. Therefore, the following literature review discusses the general field of damage mechanics, whereas developments specifically related to laminated composites are discussed in more detail in the introduction to Part II.

The research fields of fracture mechanics and damage mechanics are often related and in some cases contain significant commonality. For the purpose of the current research we define fracture mechanics to be that branch of mechanics wherein a crack is treated as a boundary of the body of interest, whereas damage mechanics is considered to be that branch of mechanics wherein the effects of cracks are included in constitutive equations rather than in boundary conditions. The usefulness of damage mechanics is apparent when one considers a body containing numerous microcracks for which an exact analytic solution is often untenable. Since in many cases internal cracking is noncatastrophic, it is pragmatic to consider the locally averaged effect of the cracks on the response of the body. This approach was first utilized by Kachanov in 1958[1]. Since that time the field of damage

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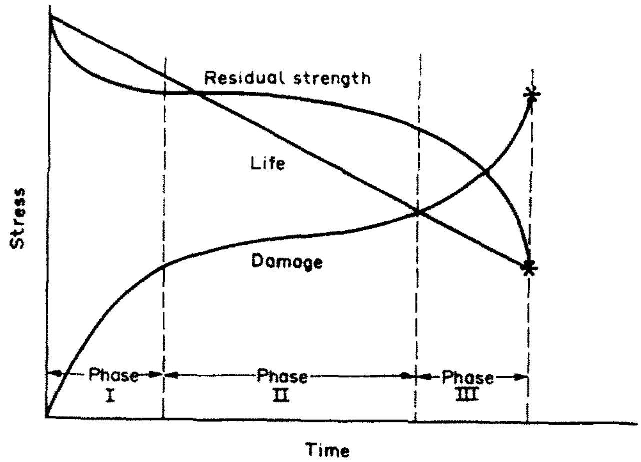

Fig. 1. Damage accumulation in a continuous fiber composite subjected to monotonic load or strain controlled cyclic fatigue [8].

mechanics has grown rapidly to the current state of development[2]. However, the predominant body of research to date has centered on the application of the method to statistically isotropic media.

Microcrack damage has been observed in a wide variety of media, including metals[3], concrete[4], geologic media[5], and composites[6–12]. The significance of this damage lies in the fact that numerous global material properties such as stiffness, damping and residual strength may be substantially altered during the life of the component, as shown in Fig. 1[13].

Attempts to model damage initially were somewhat phenomenological in nature[1, 3]. However, considerable research has shown that this approach can often be justified by micromechanics[14–18] for initially isotropic materials[2]. Fracture based concepts have recently been utilized to model damage development[19–22]. Although the first of these studies[19] contains a general theory which may be applied to fibrous composites, it has so far only been utilized for quasi-isotropic random particulate composites such as solid rocket propellant[20], and as such has not been applied to continuous fiber composites. The theory in the latter two[21, 22] has been utilized to develop fatigue matrix crack growth laws for laminated composites. Kachanov's technique[1] has also been applied to fibrous composites[23] and although promising results were obtained, the model was utilized in uniaxial form only.

The concept of damage as an internal state variable has been previously utilized in continuum mechanics/thermodynamics based theories for crystalline and/or brittle materials[24–31], as well as for non-linear viscoelastic materials[18]. A study has been made of the effect of vector-valued damage parameters on various compliance terms[32], and this methodology is currently undergoing further development[33, 34].

The foregoing discussion indicates that important progress has been made in characterizing damage in a variety of media. However, with a few notable exceptions[16, 21–23, 25, 35–38], applications have been made only to initially isotropic media. Therefore, it is the contention of these authors that substantial and continued research is warranted to develop a model of damage in laminated continuous fiber composites. In this paper an attempt will be made to utilize many of the concepts embodied in the previously referenced research efforts to develop a thermomechanical constitutive model for damage in composites which is rigorously based in continuum mechanics/thermodynamics and is generic with regard to material type, load spectrum, and specimen geometry.

The model will utilize the concept of a local volume element with statistically homogeneous damage to construct constitutive equations relating stress, strain, and damage. Unlike methods which model the local volume analytically (called micromechanics), the current research will model the local volume element experimentally (called phenomenological). The model will therefore not be restricted to linear elastic media with homo-

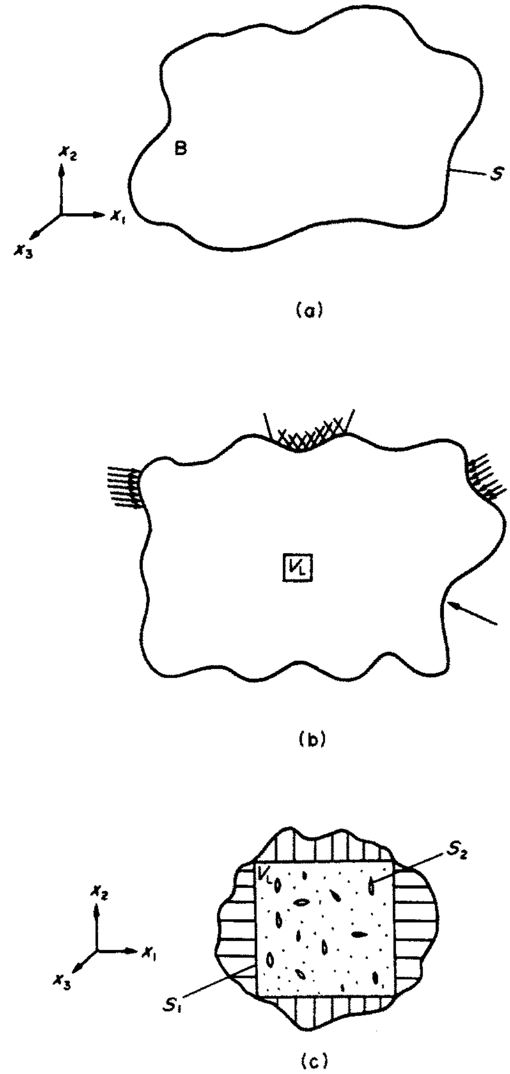

Fig. 2. Structural component labelled B: (a) undamaged state; (b) with applied tractions; (c) local volume element.

geneous elastic properties. Furthermore, the model will be applicable to cracks which are oriented and of heterogeneous and irregular size and shape. The effect of the cracks will be reflected through locally averaged quantities describing the kinematics of the cracks. The output of the model will be a set of constitutive equations which apply on a scale that is small compared to the boundary value problem of interest. Therefore, it will be applicable to the analysis of bodies with stress gradients and heterogeneous damage states.

CHARACTERIZATION OF DAMAGE AS A SET OF INTERNAL STATE VARIABLES

Consider an initially unloaded and undamaged composite structural component, denoted B, as shown in Fig. 2(a), where undamaged is defined here to mean that the body may be considered to be continuous (without cracks) on a scale several orders of magnitude smaller than the smallest external dimension of the component. Although cracks may exist in the initial state, their total surface area is assumed to be small compared to the external surface area of the component. Under this assumption the body is assumed to be simply connected and we call the initial bounding surface the external boundary S. Although the component is undamaged, there may exist local heterogeneity caused by processing and second phase materials including fibers, matrix tougheners and voids. In addition, the body may be subjected to some residual stress state due to processing, cool down, etc.

Now suppose that the component is subjected to some traction and/or deformation history, as shown in Fig. 2(b). The specimen will undergo a thermodynamic process which will in general be in some measure irreversible. This irreversibility is introduced by the occurrence of such phenomena as material inelasticity (even in the absence of damage), fracture (both micro- and macroscale), friction (due to rubbing and/or slapping of fractured surfaces), temperature flux, and chemical change. While all of these phenomena can and do commonly occur in composites, in the present research it will be assumed that all irreversible phenomena of significance occur in small zones near crack surfaces. Outside these zones, the behavior will be considered to be elastic and therefore reversible under constant temperature conditions. All fracture events will be termed damage. Due to these fracture events, the body will necessarily become multiply connected, and all newly created surfaces not intersecting the external boundary will be termed internal boundaries. Because of the above assumptions the model may be limited to polymeric and ceramic matrix composites at temperatures well below the glass transition temperature T_g or melting temperature, where viscoelasticity in matrix materials is small. Metal matrix composites may have to be excluded due to complex post-yielded behavior of the matrix.

While fracture involves changes in the boundary conditions governing a complex field problem, it is hypothesized that one may neglect boundary condition changes caused by creation and alteration of both internal and external surfaces created during fracture as long as the resulting damage in the specimen is statistically homogeneous on a local scale which is small compared to the scale of the body of interest. However, the total newly created surface area (which includes internal surfaces) may be large compared to the original external surface area. Under the condition of small-scale statistical homogeneity all continuum based conservation laws are assumed to be valid on a global scale in the sense that all changes in the continuum problem resulting from internal damage are reflected only through alterations in constitutive behavior. Typical microstructural events which may qualify as damage are matrix cracking in lamina, fiber–matrix debonding, localized interlaminar delamination and fiber fracture. Large-scale changes in the external surface such as edge delaminations, however, are treated as boundary effects which must be reflected in conservation laws via changes in the external boundary conditions rather than in constitutive equations[36, 39].

THERMODYNAMICS OF MEDIA WITH DAMAGE

We now proceed to construct a concise model of the composite with damage. To do this, consider once again the structural component, denoted B in Fig. 2(a). The body B is assumed to be of the scale of some appropriate boundary value problem of interest. Now consider some local element labelled V_L and with external surface faces S_1 arbitrarily chosen normal to a set of Cartesian coordinate axes (x_1, x_2, x_3), as shown in Fig. 2(c). The element V_L extracted from B and the newly created surfaces, denoted S_2 and with volume V_c, are subjected to appropriate boundary conditions so that the element response is identical to that when it is in B. Furthermore, the volume of the element is defined to be V_L, which includes the volume of any initial voids. The scale of V_L is chosen so that its dimensions are small compared to the dimensions of B, but at the same time, the dimensions of V_L are large enough to guarantee statistical homogeneity of the material heterogeneities and defects in V_L even though the total surface area of defects may be of the same order of magnitude as S_1[40]. Suppose furthermore that in the absence of defects or at constant damage state the material behavior is linearly thermoelastic. Now consider the local volume element V_L. For the case where tractions or displacements are applied uniformly to the external boundary of V_L, the average stresses and strains in V_L will be determinable from the external boundary tractions or displacements.

Although the damage process actually involves the conversion of strain energy to surface energy, the fact that the damage is reflected in the local constitutive equations rather than boundary conditions suggests that it be treated as a set of energy dissipative internal state variables which are not discernible on the external boundary of the local element.

Review of thermodynamic constraints on linear thermoelastic media

The following notation is adopted. Quantities without capitalized subscripts denote pointwise quantities. Those with subscripts L denote quantities which are averaged over the local element V_L. Finally, subscript E denotes linear thermoelastic properties.

Under the conditions described in the previous section the pointwise Helmholtz free energy per unit volume h of the undamaged linear elastic medium may be expressed as a second-order expansion in terms of strain ε_{ij} and temperature T as follows[41]:

$$h \equiv u - Ts = h(\varepsilon_{ij}, T) = A + B_{ij}\varepsilon_{ij} + \tfrac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + D\Delta T + E_{ij}\varepsilon_{ij}\Delta T + \tfrac{1}{2}F\Delta T^2 \quad (1)$$

where u and s are the internal energy and entropy per unit volume, respectively, and A, B_{ij}, C_{ijkl}, D, E_{ij} and F are material parameters which are independent of strain and temperature and $\Delta T \equiv T - T_R$, where T_R is the reference temperature at which the strains are zero at zero external loads. In addition, we assume here that all motions are associated with small deformations. Furthermore, inertial effects and electromagnetic coupling are assumed to be negligible.

Pointwise conservation laws appropriate to the body are given below.

(1) Conservation of linear momentum

$$\sigma_{ji,j} = 0 \quad (2)$$

where σ_{ij} is the work conjugate stress tensor to the strain tensor ε_{ij} and body forces are assumed to be negligible.

(2) Conservation of angular momentum (assuming body moments may be neglected)

$$\sigma_{ij} = \sigma_{ji}. \quad (3)$$

(3) Balance of energy

$$\dot{u} - \sigma_{ij}\dot{\varepsilon}_{ij} + q_{j,j} = r \quad (4)$$

where q_j are the components of the heat flux vector, and r is the heat source per unit volume. In addition, dots denote time differentiation and $_{,j} \equiv \partial/\partial x_j$.

(4) The second law of thermodynamics

$$\dot{s} - \frac{r}{T} + \left(\frac{q_j}{T}\right)_{,j} \geq 0. \quad (5)$$

Furthermore

$$\varepsilon_{ij} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad (6)$$

where u_i are the components of the displacement vector. Constraints imposed by the second law of thermodynamics will result in[41]

$$s = s_E = -\frac{\partial h_E}{\partial T} = -D - E_{ij}\varepsilon_{ij} - F\Delta T \quad (7)$$

and

$$\sigma_{ij} = \sigma_{Eij} = \frac{\partial h_E}{\partial \varepsilon_{ij}} = B_{ij} + C_{ijkl}\varepsilon_{kl} + E_{ij}\Delta T \quad (8)$$

where B_{ij} are interpreted as components of residual stresses at the reference temperature at which $\Delta T = 0$, and[41]

$$q_i \cong -k_{ij} g_j \tag{9}$$

where

$$g_j \equiv T_{,j} \tag{10}$$

and k_{ij} is the thermal conductivity tensor.

Thermodynamic constraints with local damage

It is our intention to construct locally averaged field equations which are similar in form to the pointwise field equations discussed above. In performing this averaging process the pointwise Helmholtz free energy described in eqn (1) will undergo a natural modification to include the energy conversion due to crack formation.

Now consider the local element shown in Fig. 2(c) with traction boundary conditions on the external surface S_1. In addition, the interior of V_L is assumed to be composed entirely of linear elastic material and cracks (which may include thin surface layers of damage). Integrating pointwise eqns (1)–(6) over the local volume will result in

$$h_{EL} = A_L + B_{Lij}\varepsilon_{Lij} + \tfrac{1}{2}C_{Lijkl}\varepsilon_{Lij}\varepsilon_{Lkl} + D_L \Delta T_L + E_{Lij}\varepsilon_{Lij}\Delta T_L + \tfrac{1}{2}F_L \Delta T_L^2 \tag{11}$$

where A_L, B_{Lij}, C_{Lijkl}, D_L, E_{Lij}, and F_L are locally averaged material constants. Also

$$\sigma_{Lji,j} = 0 \tag{12}$$

$$\sigma_{Lij} = \sigma_{Lji} \tag{13}$$

$$\dot{u}'_L - \sigma_{Lij}\dot{\varepsilon}_{Lij} + q_{Lj,j} = r_L \tag{14}$$

and

$$\dot{s}_L - \frac{r_L}{T_L} + \left(\frac{q_{Lj}}{T}\right)_{,j} \geqslant 0 \tag{15}$$

where u'_L, called the effective local internal energy, is given by

$$\dot{u}'_L \equiv \dot{u}_{EL} + \dot{u}^c_L \tag{16}$$

u_{EL} represents the internal energy of the equivalent uncracked body, given by

$$\dot{u}_{EL} \equiv \frac{1}{V_L}\int_{V_L} \dot{u}\, dV - \frac{1}{V_L}\int_{S_2} T^E_i \dot{u}_i\, dS \tag{17}$$

where T^E_i are called equivalent tractions, representing tractions in the uncracked body acting along fictitious crack faces, as described in detail in the Appendix, and $\dot{u}^c_L$ is the mechanical power output due to cracking, given by

$$\dot{u}^c_L \equiv -\frac{1}{V_L}\int_{S_2} T^c_i \dot{u}_i\, dS \tag{18}$$

where T^c_i are fictitious tractions applied to the crack faces which represent the difference between the actual crack face tractions and T^E_i. Furthermore, the locally averaged stress is given by

$$\sigma_{Lij} \equiv \frac{1}{V_L} \int_{V_L} \sigma_{ij} \, dV \tag{19}$$

and the locally averaged strain is given by

$$\varepsilon_{Lij} \equiv \frac{1}{V_L} \int_{S_1} \frac{1}{2}(u_i n_j + u_j n_i) \, dS \tag{20}$$

where n_i are components of the unit outer normal vector to the surface S_1. Equations (11)–(15) are identical in form to eqns (1)–(5), respectively. Further details on this similarity are given in the Appendix.

On the basis of this similarity we now define the locally averaged Helmholtz free energy[19, 39]

$$h_L \equiv u'_L - T_L s_L = u_{EL} - T_L s_L + u^c_L = h_{EL} + u^c_L \tag{21}$$

where it can be seen from definition (17) that h_{EL} is the locally averaged elastic Helmholtz free energy for which residual damage is zero.

The similarity between the pointwise and local field equations leads to the conclusion that

$$s_L = -\frac{\partial h_L}{\partial T_L} \tag{22}$$

$$\sigma_{Lij} = \frac{\partial h_L}{\partial \varepsilon_{Lij}} = \frac{\partial h_{EL}}{\partial \varepsilon_{Lij}} + \frac{\partial u^c_L}{\partial \varepsilon_{Lij}} \tag{23}$$

$$q_{Li} \cong -k_{Lij} g_{Lj} \tag{24}$$

and

$$g_{Lj} \equiv T_{L,j} \tag{25}$$

where

$$k_{Lik} \equiv \frac{1}{g_{Lk} V_L} \int_{V_L} k_{ij} g_j \, dV. \tag{26}$$

Note the similarity between eqns (7)–(10) and (22)–(25), respectively.

Equations (23) will serve as the basis for thermomechanical stress–strain relations in damaged composites. All damage will be reflected through the local energy due to cracking u^c_L. This term will be modelled with internal state variables characterizing the various damage modes.

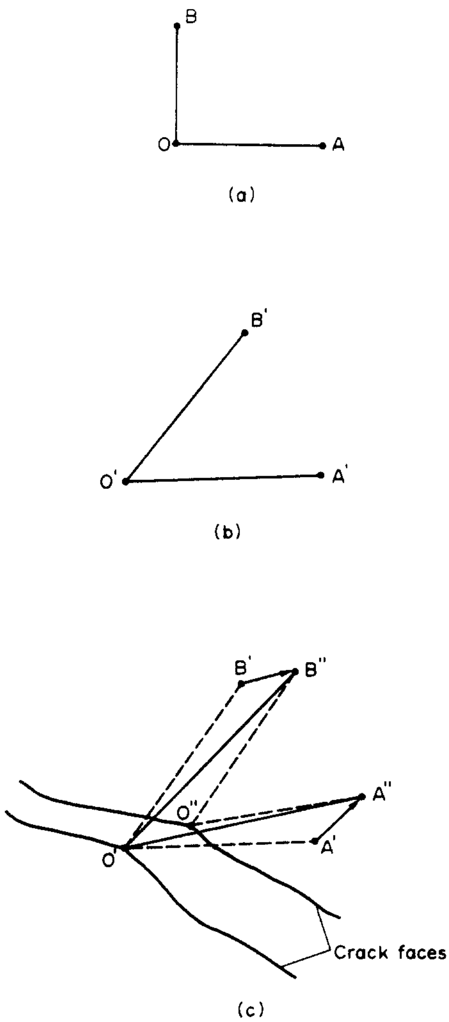

Fig. 3. Kinematics of the damage process: (a) point "O" prior to deformation; (b) point "O" after deformation and prior to fracture; (c) point "O" after fracture.

Description of the internal state

In order to describe the internal state, we first consider the kinematics of a typical point O with neighboring points A and B, as shown in Fig. 3. Before deformation lines OA and OB are orthogonal, as shown in Fig. 3(a). After deformation we imagine that lines joining O', A', and B' are as shown in Fig. 3(b), and just at the instant that deformation is completed, a crack forms normal to the plane of AOB through point O', as shown in Fig. 3(c). Furthermore, point O' becomes two material points O' and O'' on opposite crack faces and points A' and B' deform further to points A'' and B''. It is assumed that all displacements, including displacement jumps across crack faces, are infinitesimal, so that strain gages attached at points O, A, and B record only the deformation $A''O'B''$. However, the actual strain is associated with $A''O''B''$. Therefore, it is essential to construct an internal state variable which will relate these two strain descriptions. We therefore construct the vectors $\mathbf{u}^c$ connecting O' and O'' and $\mathbf{n}^c$ describing the normal to the crack face at O', as shown in Fig. 3(c). It should be noted that $\mathbf{u}^c$ can be used to construct a pseudo-strain representing the difference in rotation and extension of lines $A''O'B''$ and $A''O''B''$.

Now recall that the mechanical power output during cracking is given by eqn (18). We assume that at any point in time t_1 tractions T_i can be applied along the crack faces which will result in an energy equivalent to that produced by the damage process

$$u_L^c(t_1) = -\frac{1}{V_L}\int_{S_2(t_1)} T_i u_i^c \, dS. \tag{27}$$

The quantities T_i do not necessarily coincide with the terms in the integrand of eqn (18) since the process is in some measure irreversible. However, we define them such that the total energies in eqns (18) and (27) are equivalent. For convenience we will call them crack closure tractions, although they do not necessarily result in complete crack closure.

Guided by the fact that $\mathbf{u}^c$ and $\mathbf{n}^c$ describe the kinematics of the cracking process at point O, we now define the following second-order tensor valued internal state variable:

$$\alpha_{ij} \equiv u_i^c n_j^c \Rightarrow [\alpha_{ij}] = \begin{bmatrix} u_1^c n_1^c & u_1^c n_2^c & u_1^c n_3^c \\ u_2^c n_1^c & u_2^c n_2^c & u_2^c n_3^c \\ u_3^c n_1^c & u_3^c n_2^c & u_3^c n_3^c \end{bmatrix}. \tag{28}$$

The above description has been previously proposed by Kachanov[42]. Substituting the above into eqn (27) and utilizing Cauchy's formula gives

$$u_L^c = -\frac{1}{V_L}\int_{S_2} \sigma_{ij}^c \alpha_{ij} \, dS \tag{29}$$

where it should be pointed out that integration is performed with respect to undeformed coordinates.

Note that the components of $\mathbf{u}^c$ can be recovered from eqns (28) by using simple row multiplication on α_{ij}

$$u_i^2 = u_i^c n_j^c u_i^c n_j^c \qquad \text{(no sum on } i). \tag{30}$$

Similarly, $\mathbf{n}^c$ can be recovered by using column multiplication on α_{ij}

$$n_i^2 = u_i^c n_j^c u_i^c n_j^c / (\mathbf{u}^c)^2 \qquad \text{(no sum on } j). \tag{31}$$

Therefore, although it would not be necessary to actually perform the operations described in eqns (30) and (31), the normal and shear modes of crack displacement can be recovered from α_{ij}.

Note furthermore that α_{ij} is generally an asymmetric tensor, and that a symmetric alternative to eqns (28) could not be utilized to recover normal and shear modes as described in eqns (30) and (31). As an example, consider the following decomposition of eqns (28) into symmetric and anti-symmetric components:

$$\alpha_{ij} = \omega_{1ij} + \omega_{2ij} \tag{32}$$

where

$$\omega_{1ij} \equiv \tfrac{1}{2}(u_i^c n_j^c + u_j^c n_i^c) \tag{33}$$

and

$$\omega_{2ij} \equiv \tfrac{1}{2}(u_i^c n_j^c - u_j^c n_i^c). \tag{34}$$

In order for the anti-symmetric tensor ω_{2ij} to be zero, $\mathbf{u}^c$ and $\mathbf{n}^c$ must be parallel vectors, implying pure mode I fracture. In this case ω_{1ij} could be decomposed into a vector (in local coordinates), thus resulting in vector-valued internal state variables. For the case where the cracks in the local volume V_L are randomly oriented and of statistically homogeneous shape and size, the surface integral in eqn (29) may be carried out over all cracks. However, if various groups of cracks in the local volume V_L are distinguished by markedly different crack normals $\mathbf{n}^c$ or geometries, then it will be necessary to distinguish between the damage modes in order to retain the kinematic features of the damage process. Therefore, define the locally averaged internal state variable α^{η}_{Lij} for the ηth damage mode as follows:

$$\alpha^{\eta}_{Lij} \equiv \frac{1}{V_L}\int_{S_2^{\eta}} u_i^c n_j^c \, \mathrm{d}S = \frac{1}{V_L}\int_{S_2^{\eta}} \alpha_{ij} \, \mathrm{d}S \tag{35}$$

where

$$S_2 = \sum_{\eta=1}^{N} S_2^{\eta} \tag{36}$$

and N is the number of damage modes. For a continuous fiber laminated composite, the modes might be represented by matrix cracks, interply delamination, fiber fracture, and fiber–matrix debond ($N = 4$). For a quasi-isotropic chopped-fiber metal matrix composite, a single isotropic damage tensor might suffice for randomly oriented matrix cracking ($N = 1$).

Therefore, if we define $\sigma^{c\eta}_{Lij}$ to be the average crack closure stress for the ηth damage mode such that

$$\sigma^{c\eta}_{Lkl}\alpha^{\eta}_{Lkl} \equiv \frac{1}{V_L}\int_{S_2} \sigma^c_{ij}\alpha_{ij} \, \mathrm{d}S \tag{37}$$

it follows from eqns (29) and (35)–(37) that

$$u^c_L = -\sigma^{c\eta}_{Lij}\alpha^{\eta}_{Lij} \tag{38}$$

where we have assumed that repeated indices η imply summation over the range N. It is clear from the above discussion that the value of N must be sufficiently large to recover the essential physics of the damage process. In a mathematical sense, this implies that, whereas the mapping from α_{ij} to α^{η}_{Lij} is unique, the inverse should also be true in an approximate sense. However, there is no clearcut definition for the range N which will lead to an accurate description of the internal damage state. Note also that both u_i and n_j in eqns (35) will be affected by crack interaction in the local volume.

As an example, consider the case of mode I opening of an elliptic crack. For this case, eqn (35) will result in dependence of α^{η}_{Lij} on the volume of the inclusion. Although analytic models for linear elastic bodies with cracks result in response which is dependent on the surface area of cracks only[15–17], it should be pointed out that they also require the average crack diameter. This quantity is replaced herein by the crack opening displacement, which is proportional to the crack diameter in a linear elastic body. Therefore, specifying the crack opening displacement is equivalent to specifying the crack diameter.

Now consider eqn (38) in further detail. The kinetic quantities $\sigma^{c\eta}_{Lij}$ may be interpreted as generalized stresses which are energy conjugates to the kinematic strain-like internal state variables α^{η}_{Lij}. We infer from this that there exists a constitutive relation between these variables of the form

$$\sigma^{c\eta}_{Lij} = \sigma^{c\eta}_{Lij}(\varepsilon_{Lkl}, T_L, \alpha^{\mu}_{Lkl}) \tag{39}$$

which is history dependent via the explicit dependence on the internal state variables.

Therefore, substituting eqn (39) into eqn (38) will give

$$u^c_L(t_1) = \int_{-\infty}^{t_1} \dot{u}^c_L(t)\,dt = u^c_L(\varepsilon_{Lkl}(t_1),\ T_L(t_1),\ \alpha^{\mu}_{Lkl}(t_1)). \tag{40}$$

It is now proposed that u^c_L be expanded in a Taylor series which is second order in each of the arguments in eqn (40) as follows:

$$\begin{aligned} u^c_L = {} & G^{\eta}_{ij}\alpha^{\eta}_{Lij} + H^{\eta}_{ij}\alpha^{\eta}_{Lij}\Delta T_L + I^{\eta}_{ijkl}\varepsilon_{Lij}\alpha^{\eta}_{Lkl} + J^{\eta\zeta}_{ijkl}\alpha^{\eta}_{Lij}\alpha^{\zeta}_{Lkl}\Delta T_L \\ & + L^{\eta\zeta}_{ijklmn}\varepsilon_{Lij}\alpha^{\eta}_{Lkl}\alpha^{\zeta}_{Lmn} + \tfrac{1}{2}M^{\eta}_{ijklmn}\varepsilon_{Lij}\varepsilon_{Lkl}\alpha^{\eta}_{Lmn} \\ & + N^{\eta}_{ijkl}\varepsilon_{Lij}\alpha^{\eta}_{Lkl}\Delta T_L + P^{\eta}_{ij}\alpha^{\eta}_{Lij}\Delta T^2_L + \tfrac{1}{2}Q^{\eta\zeta}_{ijklmnpq}\varepsilon_{Lij}\varepsilon_{Lkl}\alpha^{\eta}_{Lmn}\alpha^{\zeta}_{Lpq} \\ & + R^{\eta\zeta}_{ijklmn}\varepsilon_{Lij}\alpha^{\eta}_{Lkl}\alpha^{\zeta}_{Lmn}\Delta T_L + S^{\eta\zeta}_{ijkl}\alpha^{\eta}_{Lij}\alpha^{\zeta}_{Lkl}\Delta T^2_L \\ & + T^{\eta}_{ijklmn}\varepsilon_{Lij}\varepsilon_{Lkl}\alpha^{\eta}_{Lmn}\Delta T_L + U^{\eta}_{ijkl}\varepsilon_{Lij}\alpha^{\eta}_{Lkl}\Delta T^2_L \\ & + V^{\eta\zeta}_{ijklmnop}\varepsilon_{Lij}\varepsilon_{Lkl}\alpha^{\eta}_{Lmn}\alpha^{\zeta}_{Lop}\Delta T_L + W^{\eta}_{ijklmn}\varepsilon_{Lij}\varepsilon_{Lkl}\alpha^{\eta}_{Lmn}\Delta T^2_L \\ & + X^{\eta\zeta}_{ijklmn}\varepsilon_{Lij}\alpha^{\eta}_{Lkl}\alpha^{\zeta}_{Lmn}\Delta T^2_L + \tfrac{1}{2}Y^{\eta\zeta}_{ijklmn}\varepsilon_{Lij}\alpha^{\eta}_{Lkl}\alpha^{\zeta}_{Lmn}\Delta T_L \\ & + Z^{\eta\zeta}_{ijklmnpq}\varepsilon_{Lij}\varepsilon_{Lkl}\alpha^{\eta}_{Lmn}\alpha^{\zeta}_{Lpq}\Delta T^2_L \end{aligned} \tag{41}$$

where all terms are at least linear in α^{η}_{Lij} due to the fact that u^c_L depends explicitly on damage, and $\Delta T_L \equiv T_L - T_R$. Thus, substituting eqns (11) and (41) into eqns (23) and neglecting higher order terms yields

$$\sigma_{Lij} = B_{Lij} + E_{Lij}\Delta T_L + C_{Lijkl}\varepsilon_{Lkl} + I^{\eta}_{ijkl}\alpha^{\eta}_{Lkl}. \tag{42}$$

Restricting the damage to small quantities constitutes a sufficient but not a necessary condition for dropping the higher order terms. Equations (42) may be written in the following alternate form for isothermal conditions:

$$\sigma_{Lij} = \sigma^R_{Lij} + C'_{Lijkl}\varepsilon_{Lkl} \tag{43}$$

where

$$\sigma^R_{Lij} \equiv B_{Lij} \tag{44}$$

is the residual stress tensor; and

$$C'_{Lijkl}\varepsilon_{Lkl} \equiv C_{Lijkl}\varepsilon_{Lkl} + I^{\eta}_{Lijkl}\alpha^{\eta}_{Lkl} \tag{45}$$

defines the effective modulus tensor C'_{Lijkl} for any damage state. Note that although eqn (43) is similar to Kachanov's model[1], the stiffness reduction is a first-order effect of damage. Note also that the inclusion of higher order terms will result in damage dependent residual and thermal stresses, as well as non-linear stiffness loss as a function of damage.

Equation (42) is the completed description of the stress–strain relationship. Note that this equation reduces to the standard linear thermoelastic equation in the absence of damage ($\alpha^{\eta}_{Lij} = 0$).

Damage growth laws

The model is completed with the construction of the damage growth laws, which may be described in the following differential equation form:

$$\dot{\alpha}^{\eta}_{Lij} = \Omega^{\eta}_{ij}(\varepsilon_{Lkl},\ \dot{\varepsilon}_{Lkl},\ T_L,\ \alpha^{\mu}_{Lkl}) \tag{46}$$

or equivalently, when Ω^{η}_{ij} are single valued functions of time

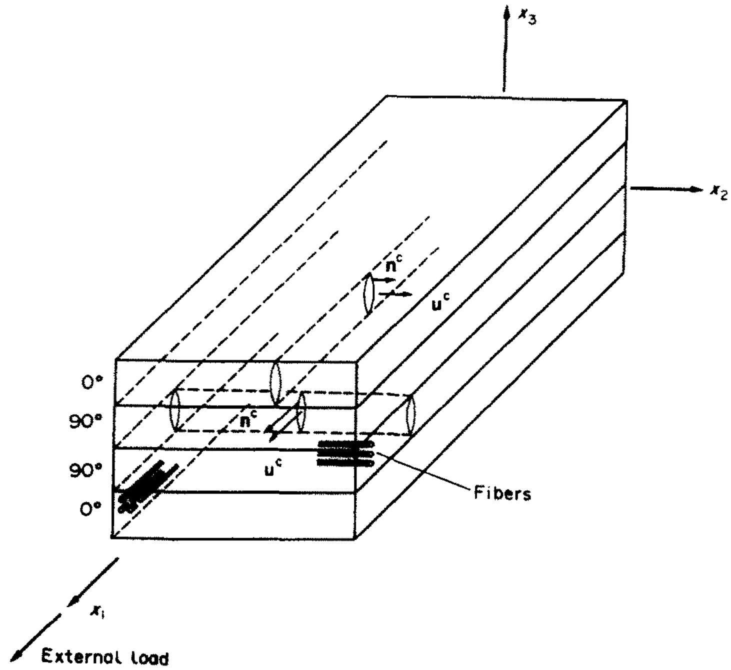

Fig. 4. Assumed damage vector directions in a $[0, 90]_s$ laminate.

$$\alpha^{\eta}_{Lij}(t_1) = \int_{-\infty}^{t_1} \Omega^{\eta}_{ij}(\varepsilon_{Lkl}(t),\ T_L(t),\ \alpha^{\mu}_{Lkl}(t))\ \mathrm{d}t. \tag{47}$$

Although the above equations are called "growth" laws they have the more general capability to model such phenomena as healing.

The precise nature of eqn (47) is determinable only through a concise experimental program coupled with an understanding of the micromechanics of the medium. Indeed, these growth laws constitute the single most complex link in the model development.

In this section an example of a first generation growth law will be constructed for predicting damage up the CDS in continuous fiber composites. Experimental evidence suggests that matrix cracks dominate the first phase of damage development in laminated composites[9–11]. Guided by this observation, a single damage tensor is considered in this section: α^1_{Lij} representing matrix cracking.

In order to completely define eqn (47), it is necessary to construct indicators of both the magnitude and direction of the damage tensor. In this first generation model it is assumed that the direction of the damage tensor is known *a priori* and does not vary as the damage state changes. Specifically, in a typical laminate, it is assumed that, for this simple example, in accordance with eqn (35), the locally averaged resultants of $\mathbf{u}^c$ and $\mathbf{n}^c$ are normal to the fiber direction in each ply, as shown in Fig. 4. Thus, for example, in a 0° ply $\alpha^1_{L22} \neq 0$, and all other components are zero, whereas in a 90° ply, $\alpha^1_{L11} \neq 0$, and all other components are zero (in global coordinates). In Part II a somewhat more general case of the damage state for matrix cracking will be considered.

Under the above assumptions, the magnitude of the damage tensor is the sole repository for history dependence in each ply. Experimental evidence indicates that for matrix cracking in randomly oriented particulate composites[43] and matrix cracks in fibrous composites[21, 22] the growth of damage surface area is related to the energy release rate G by

$$\frac{\mathrm{d}S_2}{\mathrm{d}N} \propto G^n \tag{48}$$

where S_2 represents crack area, N the number of cycles in a fatigue test, and n is some material parameter. Guided by these results, a similar law is constructed here. Equation (48) may be rewritten in the following form:

$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = KG^n \cdot \frac{\mathrm{d}N}{\mathrm{d}t} \tag{49}$$

so that it follows that

$$\dot{\alpha}^1_{L22} = \frac{\mathrm{d}\alpha^1_{L22}}{\mathrm{d}S_2} \cdot \frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{\mathrm{d}\alpha^1_{L22}}{\mathrm{d}S_2} \cdot KG^n \cdot \frac{\mathrm{d}N}{\mathrm{d}t}. \tag{50}$$

Assuming that the energy release rate is essentially mode I and therefore depends on the maximum normal strain, the damage growth law for matrix cracking is thus hypothesized to be of the form

$$\begin{aligned} \dot{\alpha}^1_{L22} &= k_1\left(\frac{\varepsilon_n - \varepsilon_{n\min}}{\alpha^1_{L22}}\right)^{n_1} \cdot \frac{\mathrm{d}\varepsilon_n}{\mathrm{d}t} && \text{if} \quad \varepsilon_{n\min} < \varepsilon_n \\ \dot{\alpha}^1_{L22} &= k_2\dot{\varepsilon}_n && \text{if} \quad \varepsilon_{n\min} \geqslant \varepsilon_n \end{aligned} \tag{51}$$

where ε_n is the local normal strain component which is normal to the fibers. Furthermore, $\varepsilon_{n\min}$ is the value of ε_n at which matrix cracking initiates. k_1, k_2, and n are experimentally determined material parameters which may depend on the initial damage state or on history dependent damage other than matrix cracks. The use of ε_n presupposes that the fracture mode is predominantly mode I in nature, which may not be the case in some complex layups. In these cases, mode II and mode III terms may be required. Note that all components of α^1_{Lij} are zero except α^1_{L22}, which is nonzero in the local ply coordinate system wherein the fibers are aligned parallel to the local x_1-axis.

Experimental evidence[44] indicates that in cross-ply laminates with multiple adjacent crossplies in sequence, it is not uncommon to observe matrix cracks which are curved rather than normal to the plane of the ply. For these cases it is necessary to carry components of α^1_{Lij} in both the x_2 and x_3 coordinate directions. Although it is hypothesized that these components may perhaps be determinable from the orientation of the maximum normal strain, ε_n, this issue is under further investigation by the authors.

Equations (51) complete the description of the damage model for the case of matrix cracking. Integration of these equations in time will lead to current values of the damage tensor which is input to constitutive eqn (42). Figure 5 shows a typical growth history for a specimen subjected to monotonically increasing deformation $u(L)$. It should be pointed out, however, that these equations may be extremely nonlinear and as such must in some cases be integrated numerically with stiff integration schemes[45].

CONCLUSION

Stress–strain relations have been developed herein which account for various forms of damage in continuous fiber composites. Furthermore, a damage growth law has been proposed for matrix cracking in fibrous composites. The model developed herein is thus a complete description necessary to characterize the thermomechanical constitution of a fibrous composite with matrix cracks (excluding failure).

The actual use of this model is complicated by the requirement for numerous experimentally determined quantities, as well as the necessity to determine locally based observable state variables by analytic methods. The construction of these parameters constitutes an entire separate research effort which is considered in Part II.

Finally, it should be pointed out that although an internal state variable growth law has been proposed herein only for matrix cracks, the model is in principle applicable to

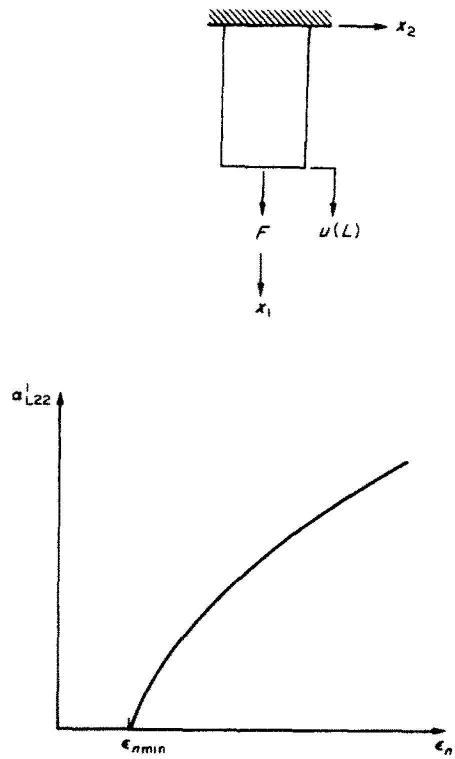

Fig. 5. Typical growth of damage in a specimen with matrix cracks.

more complex damage states in laminated composites, and research is underway to consider other damage modes[45].

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APPENDIX

Consider a local volume element with some damage state, where the crack faces are defined as traction free surfaces, as shown in Fig. A1(a1). For convenience we show only one crack, although in actuality the damage must be statistically homogeneous in V_L. Now replace the actual cracks with fictitious cracks which are described by the bounding surface between elastic and inelastic response near cracks, as shown in Fig. A1(a2). We define this surface as S_2. In order to insure that the total mechanical states in the two systems are identical, the fictitious case must include tractions labelled $T_i^{\rm F}$ on S_2.

Now suppose that $V_{\rm L}$ is subjected to boundary tractions on S_1 *in the undamaged state* as shown in Fig. A1(b1). We define an equivalent elastic problem in which the surface S_2 described in Fig. A1(a2) is cut from $V_{\rm L}$ and elastic tractions $T_i^{\rm E}$ are applied on S_2 so that the total mechanical states of the systems in Figs A1(b1) and (b2) are equivalent.

The actual system of interest is described in Fig. A1(a1). However, for pragmatic reasons we wish to replace the actual system with a fictitious system with equivalent mechanical state. To do this, we first replace Fig. A1(a1) with Fig. A1(a2), which by definition has equivalent mechanical state. Next, we define a system equivalent to Fig. A1(a2), such that

$$T_i^{\rm c} \equiv T_i^{\rm F} - T_i^{\rm E} \Rightarrow T_i^{\rm F} = T_i^{\rm c} + T_i^{\rm E} \tag{A1}$$

as shown in Fig. A1(c). Integrating the balance of energy (4) over the local volume and dividing through by the local volume results in

$$\frac{1}{V_{\rm L}}\int_{V_{\rm L}} \dot{u}\,{\rm d}V - \frac{1}{V_{\rm L}}\int_{V_{\rm L}} \sigma_{ij}\dot{\varepsilon}_{ij}\,{\rm d}V + \frac{1}{V_{\rm L}}\int_{V_{\rm L}} q_{j,j}\,{\rm d}V = \frac{1}{V_{\rm L}}\int_{V_{\rm L}} r\,{\rm d}V. \tag{A2}$$

Now consider the second term in eqn (A2). Recall that since σ_{ij} is a symmetric tensor

$$\sigma_{ij}\dot{\varepsilon}_{ij} = \tfrac{1}{2}\sigma_{ij}(\dot{u}_{i,j} + \dot{u}_{j,i}) = \sigma_{ij}\dot{u}_{i,j}. \tag{A3}$$

Thus, assuming that the stresses are negligible in $V_{\rm c}$, the volume enclosed by S_2, using the divergence theorem and substituting Cauchy's formula gives

$$\frac{1}{V_{\rm L}}\int_{V_{\rm L}} \sigma_{ij}\dot{\varepsilon}_{ij}\,{\rm d}V = \frac{1}{V_{\rm L}}\int_{V_{\rm L}-V_{\rm c}} \sigma_{ij}\dot{u}_{i,j}\,{\rm d}V = \frac{1}{V_{\rm L}}\int_{S_1} T_i\dot{u}_i\,{\rm d}S + \frac{1}{V_{\rm L}}\int_{S_2} T_i^{\rm F}\dot{u}_i\,{\rm d}S = \frac{1}{V_{\rm L}}\int_{S_1} T_i\dot{u}_i\,{\rm d}S$$
$$+ \frac{1}{V_{\rm L}}\int_{S_2} T_i^{\rm E}\dot{u}_i\,{\rm d}S + \frac{1}{V_{\rm L}}\int_{S_2} T_i^{\rm c}\dot{u}_i\,{\rm d}S \tag{A4}$$

where n_j are the components of the unit outer normal vector to the surface $S = S_1 + S_2$. Now define

$$\dot{u}_{\rm L}^{\rm c} \equiv -\frac{1}{V_{\rm L}}\int_{S_2} T_i^{\rm c}\dot{u}_i\,{\rm d}S \tag{A5}$$

which is the effective specific mechanical power output of the continuum due to the crack surface tractions. This term contains both the mechanical power due to crack extension as well as the mechanical power due to apparent stiffness loss caused by existing cracks. For the special case of a reversible process this is the time rate of change of surface energy release per unit local volume due to cracking in $V_{\rm L}$. Furthermore, define

$$\varepsilon_{{\rm L}ij} \equiv \frac{1}{V_{\rm L}}\int_{S_1} \frac{1}{2}(u_i n_j + u_j n_i)\,{\rm d}S \tag{A6}$$

and

$$\sigma_{{\rm L}ij} \equiv \frac{1}{V_{\rm L}}\int_{S_1} \frac{1}{2}(\sigma_{im}n_m x_j + \sigma_{jm}n_m x_i)\,{\rm d}S = \frac{1}{V_{\rm L}}\int_{V_{\rm L}} \sigma_{ij}\,{\rm d}V. \tag{A7}$$

Therefore, for the case of either spatially uniform surface tractions or displacements which are linear in coordinates on S_1 one readily obtains

$$\sigma_{{\rm L}ij}\dot{\varepsilon}_{{\rm L}ij} = \frac{1}{V_{\rm L}}\int_{S_1} \sigma_{ij}\dot{u}_i n_j\,{\rm d}S = \frac{1}{V_{\rm L}}\int_{S_1} T_i\dot{u}_i\,{\rm d}S. \tag{A8}$$

Although it will be assumed in the remainder of this paper that the above conditions are satisfied, they need only be approximately true if $V_{\rm L}$ is statistically homogeneous. Thus, eqn (A4) becomes

$$\frac{1}{V_{\rm L}}\int_{V_{\rm L}} \sigma_{ij}\dot{\varepsilon}_{ij}\,{\rm d}V = \sigma_{{\rm L}ij}\dot{\varepsilon}_{{\rm L}ij} + \frac{1}{V_{\rm L}}\int_{S_2} T_i^{\rm E}\dot{u}_i\,{\rm d}S - \dot{u}_{\rm L}^{\rm c}. \tag{A9}$$

Define also

$$q_{{\rm L}i,j} \equiv \frac{1}{V_{\rm L}}\int_{S_1} q_i n_j\,{\rm d}S \tag{A10}$$

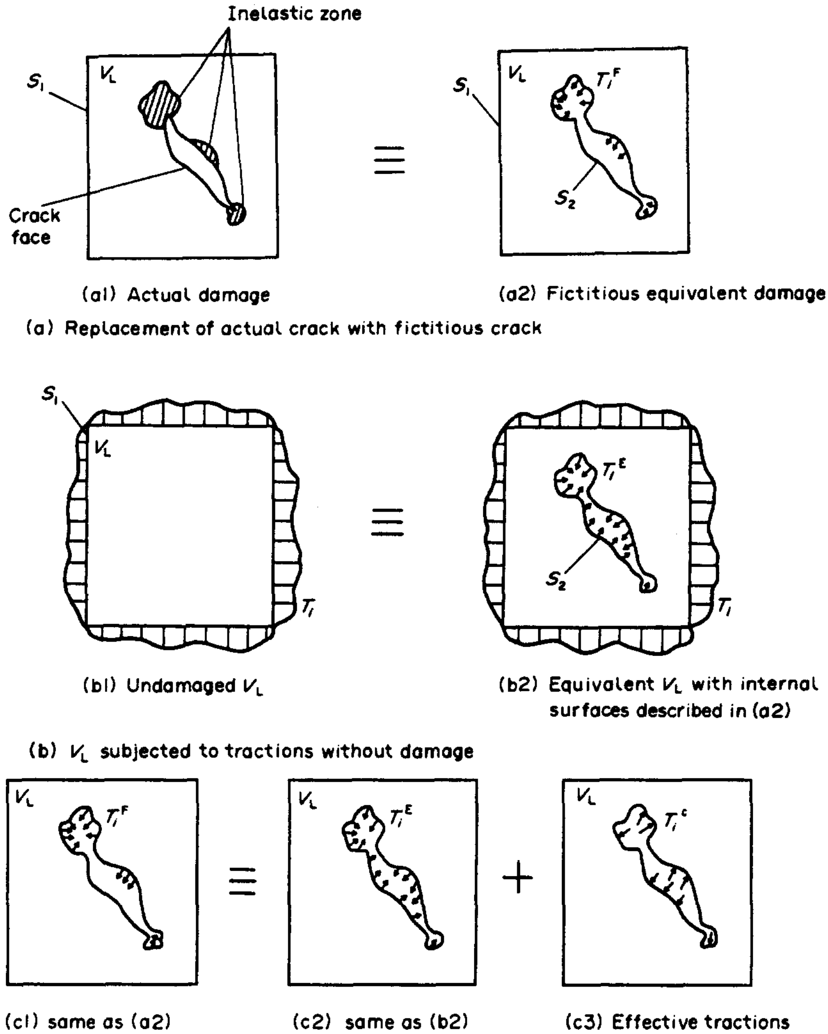

Fig. A1. Description of local volume element with damage.

and

$$r_L \equiv \frac{1}{V_L}\int_{V_L} r\,dV. \tag{A11}$$

Now define

$$\dot{u}_{EL} \equiv \frac{1}{V_L}\int_{V_L} \dot{u}\,dV - \frac{1}{V_L}\int_{S_2} T_i^E \dot{u}_i\,dS \tag{A12}$$

which can be seen from Fig. A1(b) to be the equivalent internal energy rate that would be produced in the body without cracks. Note that u_{EL} is not path dependent since it represents elastic response. Substituting eqns (A4), (A5) and (A9)–(A12) into eqn (A2) yields the following locally averaged balance of energy:

$$\dot{u}_{EL} + \dot{u}_L^c - \sigma_{Lij}\dot{\varepsilon}_{Lij} + q_{Lj,j} = r_L. \tag{A13}$$

We now define the effective internal energy u'_L (which may be path dependent) such that

$$\dot{u}'_L \equiv \dot{u}_{EL} + \dot{u}_L^c. \tag{A14}$$

Substitution of eqn (A14) into eqn (A13) results in

$$\dot{u}'_L - \sigma_{Lij}\dot{\varepsilon}_{Lij} + q_{Lj,j} = r_L \tag{A15}$$

which can be seen to be equivalent in form to energy balance law (4).

In order to construct a similar statement for entropy production inequality (5), first multiply through by T and then integrate over the local volume V_L and divide by this quantity to obtain

$$\frac{1}{V_L}\int_{V_L} \dot{s}T\,dV - \frac{1}{V_L}\int_{V_L} r\,dV + \frac{1}{V_L}\int_{V_L} T(q_j/T)_{,j}\,dV \geq 0. \tag{A16}$$

Now define

$$T_L \equiv \frac{1}{V_L}\int_{V_L} T\,dV \tag{A17}$$

and

$$\dot{s}_L \equiv \frac{1}{T_L V_L}\int_{V_L} \dot{s}T\,dV \tag{A18}$$

so that substitution of definitions (A11), (A17) and (A18) into (A16) will result in

$$\dot{s}_L T_L - r_L + \frac{1}{V_L}\int_{V_L} T(q_j/T)_{,j}\,dV \geq 0. \tag{A19}$$

Now note that the last term in inequality (A19) may be written as follows using the product rule:

$$\frac{1}{V_L}\int_{V_L} T(q_j/T)_{,j}\,dV = \frac{1}{V_L}\int_{V_L} q_{j,j}\,dV - \frac{1}{V_L}\int_{V_L} (q_j g_j/T)\,dV. \tag{A20}$$

Define now

$$T_{L,j} \equiv \frac{1}{V_L}\int_{S_L} T n_j\,dS. \tag{A21}$$

Thus, for the case when T is a linear function of coordinates in V_L, definitions (A10) and (A21) may be substituted into eqn (A20) and this result into inequality (A19) to obtain

$$\dot{s}_L - \frac{r_L}{T_L} + \left(\frac{q_{Lj}}{T_L}\right)_{,j} \geq \dot{s}_c \geq 0 \tag{A22}$$

where

$$\dot{s}_c \equiv (1/T_L V_L)\int_{V_L} (q_j g_j/T)\,dV - (1/T_L^2 V_L^2)\int_{V_L} q_j\,dV \cdot \frac{1}{V_L}\int_{V_L} g_j\,dV. \tag{A23}$$

$\dot{s}_c$ can be shown to be strictly nonnegative with the assumption that T is everywhere nonnegative, along with eqn (9).

We now assume that the local volume is small enough compared to B that the standard procedure may be utilized to obtain the linear conservation of momentum equations[40]

$$\sigma_{Lji,j} = 0 \tag{A24}$$

similar to pointwise eqn (2), and the conservation of angular momentum may also be obtained

$$\sigma_{Lij} = \sigma_{Lji} \tag{A25}$$

similar to eqn (3). Thus, it is assumed that no body moments are introduced via material inhomogeneity or other sources. This assumption must be relaxed when the model is utilized for interply delamination, since in this case the local volume element goes through the entire laminate thickness.

Equations (A24), (A25), (A15), (A22), (A14), (A18), (A5), (A7), and (A6) are rewritten as eqns (12)–(20), respectively, in the main text.